3.12) a) $V = \mathbb{R}^3$

$$S = \{[x_1\ x_2\ x_3]^T \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$$

Busco gen de S:

$x_1 + x_2 + x_3 = 0 \to x_1 = -x_2 - x_3$

$\to \bar{X} = (-x_2 - x_3, x_2, x_3) = x_2.(-1,1,0) + x_3.(-1,0,1)$

Como son LI y generan S $\to B_S = \{\underbrace{(-1,1,0)}_{v_1}, \underbrace{(-1,0,1)}_{v_2}\}$.

$$S^\perp = \{v \in \mathbb{R}^3 : \underbrace{(v_1, v) = 0}_{\text{I}} \wedge \underbrace{(v_2, v) = 0}_{\text{II}}\}$$

I $\to$ siendo $v = (x,y,z) \to (-1,1,0).(x,y,z) = 0$

$\to -x + y = 0 \to x = y$ Primera condición

II $\to (-1,0,1).(x,y,z) = 0 \to -x + z = 0 \to x = z$ segunda cond.

$\to S^\perp = \{v \in \mathbb{R}^3 : x = y \wedge x = z\}$

entonces los v que cumplen son los que tienen la forma:

$\bar{X} = (x, x, x) = x.(1,1,1)$

Por lo tanto una base de $S^\perp$ es:

$$\boxed{B_{S^\perp} = \{(1,1,1)\}}$$

Ahora junto las bases:

$B_S \cup B_{S^\perp} = \{\underbrace{(-1,1,0),(-1,0,1),(1,1,1)}_{LI}\} \to$ Base de $\mathbb{R}^3$

$\to \underbrace{(x,y,z)}_{v} = \underbrace{\alpha.(-1,1,0) + \beta.(-1,0,1)}_{v_S} + \underbrace{\gamma.(1,1,1)}_{v_{S^\perp}}$

Entonc.:

$$\begin{cases} -\alpha-\beta+\gamma = x \rightarrow -y+2\gamma-\beta = x \rightarrow \beta = -y+2\gamma-x \rightarrow \boxed{\beta = -\frac{x}{3}-\frac{y}{3}+\frac{2}{3}z} \\ \alpha+\gamma = y \rightarrow \alpha = y-\gamma \rightarrow \boxed{\alpha = -\frac{x}{3}+\frac{2}{3}y-\frac{z}{3}} \\ \beta+\gamma = z \rightarrow -y+3\gamma-x = z \rightarrow \gamma = \frac{z+x+y}{3} = \boxed{\frac{x}{3}+\frac{y}{3}+\frac{z}{3} = \gamma} \end{cases}$$

Por lo tanto:

$$\underbrace{(x,y,z)}_{v\in\mathbb{R}^3} = \underbrace{\left(\frac{-x+2y-z}{3}\right).(-1,1,0)+\left(\frac{-x-y+2z}{3}\right).(-1,0,1)}_{v_S\in S}+\underbrace{\left(\frac{x+y+z}{3}\right).(1,1,1)}_{v_{S^\perp}\in S^\perp}$$

6) $V = \mathbb{R}^4$

$$S = \left\{ [x_1\ x_2\ x_3\ x_4]^T \in \mathbb{R}^4 : \begin{cases} x_1+x_2=0 \\ x_3-2x_4=0 \end{cases} \right\}$$

Busco gen. de S:

$$\begin{cases} x_1+x_2=0 \rightarrow x_1=-x_2 \\ x_3-2x_4=0 \rightarrow x_3=2x_4 \end{cases}$$

$$\rightarrow \bar{X} = (-x_2, x_2, 2x_4, x_4) = x_2.(-1,1,0,0)+x_4.(0,0,2,1)$$

Como son LI y generan S:

$$B_S = \{\underbrace{(-1,1,0,0)}_{v_1}, \underbrace{(0,0,2,1)}_{v_2}\}$$

$$S^\perp = \{ v\in\mathbb{R}^4 : \underbrace{(v_1,v)=0}_{\text{I}} \wedge \underbrace{(v_2,v)=0}_{\text{II}} \}$$

(I) $\rightarrow$ sea $v=(x_1,x_2,x_3,x_4) \rightarrow (-1,1,0,0).(x_1,x_2,x_3,x_4)=0$

$\rightarrow -x_1+x_2=0 \rightarrow x_1=x_2$ 1ra condición

II $\to (0,0,2,1)\cdot(x_1,x_2,x_3,x_4) = 0 \to 2x_3 + x_4 = 0 \to x_4 = -2x_3$ 2da cond

$\to S^\perp = \{ v \in \mathbb{R}^4 : x_1 = x_2 \wedge x_4 = -2x_3 \}$

$\bar{X}$ que cumplen $\to$ $\bar{X} = (x_1, x_1, x_3, -2x_3) = x_1 \cdot (1,1,0,0) + x_3 \cdot (0,0,1,-2)$

Como son LI:

$$\boxed{B_{S^\perp} = \{(1,1,0,0),(0,0,1,-2)\}}$$

Junto bases:

$B_S \cup B_{S^\perp} = \{(-1,1,0,0),(0,0,2,1),(1,1,0,0),(0,0,1,-2)\} \to B_{\mathbb{R}^4}$.

$$\to \underbrace{(x_1,x_2,x_3,x_4)}_{v} = \underbrace{\alpha\cdot(-1,1,0,0) + \beta\cdot(0,0,2,1)}_{v_S} + \underbrace{\gamma\cdot(1,1,0,0) + \theta(0,0,1,-2)}_{v_{S^\perp}}$$

Ec.:

$$\begin{cases} -\alpha + \gamma = x_1 \to \alpha = \gamma - x_1 \to \boxed{\alpha = -\frac{x_1}{2} + \frac{x_2}{2}} \\ \alpha + \gamma = x_2 \to 2\gamma - x_1 = x_2 \to \boxed{\gamma = \frac{x_1}{2} + \frac{x_2}{2}} \\ 2\beta + \theta = x_3 \to \theta = x_3 - 2\beta \to \theta = x_3 - \frac{4}{5}x_3 - \frac{2}{5}x_4 \to \boxed{\theta = \frac{x_3}{5} - \frac{2x_4}{5}} \\ \beta - 2\theta = x_4 \to \beta - 2x_3 + 4\beta = x_4 \to 5\beta = x_4 + 2x_3 \to \boxed{\beta = \frac{2}{5}x_3 + \frac{x_4}{5}} \end{cases}$$

Por lo tanto:

~~Ahora~~

$$\underbrace{(x_1,x_2,x_3,x_4)}_{v \in \mathbb{R}^4} = \underbrace{\left(\frac{-x_1+x_2}{2}\right)\cdot(-1,1,0,0) + \left(\frac{2x_3+x_4}{5}\right)(0,0,2,1)}_{v_S \in S} + \underbrace{\left(\frac{x_1+x_2}{2}\right)\cdot(1,1,0,0) + \left(\frac{x_3-2x_4}{5}\right)(0,0,1,-2)}_{v_{S^\perp} \in S^\perp}$$

c) $V = \mathbb{C}^4$

$$S = \left\{ [x_1\ x_2\ x_3\ x_4]^T \in \mathbb{C}^4 : \begin{cases} x_1 - ix_2 + (1-i)x_3 = 0 \\ (2+i)x_2 + x_4 = 0 \end{cases} \right\}$$

Busco gen. de S:

$$\begin{cases} x_1 - ix_2 + (1-i)x_3 = 0 \to x_1 = ix_2 - (1-i)x_3 \\ (2+i)x_2 + x_4 = 0 \to x_4 = -(2+i)x_2 \end{cases}$$

$\to \bar{x} = (ix_2 - (1-i)x_3,\ x_2,\ x_3,\ -(2+i)x_2)$

$\to \bar{x} = x_2.(i,1,0,-2-i) + x_3.(-1+i,0,1,0)$

$\to B_S = \{\underbrace{(i,1,0,-2-i)}_{v_1}, \underbrace{(-1+i,0,1,0)}_{v_2}\}$

$S^\perp = \{v \in \mathbb{C}^4 : \underbrace{(v_1,v) = 0}_{\mathrm{I}} \wedge \underbrace{(v_2,v) = 0}_{\mathrm{II}}\}$

I $\to$ Sea $v = (x_1,x_2,x_3,x_4) \to (i,1,0,-2-i).(x_1,x_2,x_3,x_4) = 0$

$\to i\overline{x_1} + \overline{x_2} + (-2-i)\overline{x_4} = 0 \to i\overline{x_1} = -\overline{x_2} - (-2-i)\overline{x_4}$

$\to \overline{x_1} = \dfrac{(-\overline{x_2} - (-i-2)\overline{x_4})}{i} \cdot \dfrac{-i}{-i} \to \overline{x_1} = \dfrac{i\overline{x_2} - (-1+2i)\overline{x_4}}{1}$

$\to \overline{x_1} = i\overline{x_2} - (-1+2i)\overline{x_4} \to \boxed{x_1 = -ix_2 - (-1-2i)x_4}$

1ra condición

II $\to (-1+i,0,1,0).(x_1,x_2,x_3,x_4) = 0$

$\to (-1+i)\overline{x_1} + \overline{x_3} = 0 \to \overline{x_3} = -(-1+i)\overline{x_1}$

$\to x_3 = -(-1-i)x_1$

usando la condición:

$\to x_3 = (1+i).(-ix_2 - (-1-2i)x_4)$

$\to x_3 = -ix_2 - (-1-2i)x_4 + x_2 - (-i+2)x_4$

$\to x_3 = x_2.(-i+1) + x_4.(1+2i+i-2) \to \boxed{x_3 = x_2.(-i+1) + x_4.(-1+3i)}$

2da cond.

$\rightarrow S^{\perp} = \{ v \in \mathbb{C}^4 : x_1 = -i x_2 + (1+2i) x_4 \wedge x_3 = x_2.(1-i) + x_4.(-1+3i) \}$

$\bar{X}$ que cumplen $\rightarrow \bar{X} = (-i x_2 + (1+2i) x_4, x_2, x_2.(1-i) + x_4.(-1+3i), x_4)$

$\rightarrow \bar{X} = x_2.(-i, 1, 1-i, 0) + x_4.(1+2i, 0, -1+3i, 1)$

$\rightarrow \boxed{B_{S^{\perp}} = \{ (-i, 1, 1-i, 0), (1+2i, 0, -1+3i, 1) \}}$

Junto bases:

$B_S \cup B_{S^{\perp}} = \{ (i, 1, 0, -2-i), (-1+i, 0, 1, 0), (-i, 1, 1-i, 0), (1+2i, 0, -1+3i, 1) \} \rightarrow B_{\mathbb{C}^4}$

$\rightarrow \underbrace{(x_1, x_2, x_3, x_4)}_{v} = \underbrace{\alpha.(i, 1, 0, -2-i) + \beta.(-1+i, 0, 1, 0)}_{v_S} + \underbrace{\gamma.(-i, 1, 1-i, 0) + \theta.(1+2i, 0, -1+3i, 1)}_{v_{S^{\perp}}}$

Ec:

$$\begin{cases} \alpha i - \beta + \beta i - \gamma i + \theta(1+2i) = x_1 & \text{I} \\ \alpha + \gamma = x_2 & \text{II} \\ \beta + (1-i)\gamma + (-1+3i)\theta = x_3 & \text{III} \\ (-2-i)\alpha + \theta = x_4 & \text{IV} \end{cases}$$